BRIEF COMMUNICATIONS

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Analysis of mean momentum flux in subgrid models of turbulence

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(Received 22 April 2004; accepted 27 May 2004; published online 6 August 2004)

Using a recently proposed model spectrum for weakly sheared turbulence [M. Ishihara, K. Yoshida, and Y. Kaneda, Phys. Rev. Lett. **88**, 154501 (2002)], the mean momentum fluxes caused by subgrid scales are evaluated analytically. Using the result to calibrate the Smagorinsky coefficient, the value obtained is remarkably close to the traditional result obtained from balancing kinetic energy dissipation. For the Gaussian filter, the same conclusion is obtained when the Smagorinsky model is supplemented with the nonlinear model. It is concluded that when in practice the Smagorinsky model does not correctly predict the mean subgrid scale (SGS) stresses in large eddy simulation (LES), it must be due to either nonuniversal effects at large scales, or due to strong shear that violates the model spectrum assumed in the present analysis. Analysis of a two-parameter mixed model shows that the nonlinear term must be weighted more strongly in high shear conditions than in weak shear. © *2004 American Institute of Physics*. [DOI: 10.1063/1.1773846]

There is a fairly robust consensus that dissipation of kinetic energy of the resolved scales is the most important effect of subgrid-scale closures for LES of turbulent flow (see Refs. 1–4). Already Lilly[5] used the condition that the SGS energy dissipation $-\langle \tau_{ij}\tilde{S}_{ij}\rangle$ (where $\tau_{ij}=\widetilde{u_iu_j}-\tilde{u}_i\tilde{u}_j$ is the SGS stress tensor and $\tilde{S}_{ij}$ is the strain-rate tensor of the resolved scales) equals that predicted by the Smagorinsky model $2c_s^2\Delta^2\langle|\tilde{S}|\tilde{S}_{ij}\tilde{S}_{ij}\rangle$ (Δ being the filter scale) to derive a relationship between the Smagorinsky coefficient c_s and the universal Kolmogorov constant c_K. He obtained $c_s=(3c_K/2)^{-3/4}/\pi$ (and thus $c_s\sim 0.17$ for $c_K=1.6$) for a spectral cut-off filter. The Smagorinsky model assumes the SGS stress is perfectly correlated with the resolved strain rate which, as is well known,[2,6] does not correspond to the correlation between the real SGS stress and strain rate which in reality is much smaller. As a consequence, when the Smagorinsky coefficient is determined by Lilly's methodology, the magnitudes of the SGS stresses are generally underestimated.[3,7,8] If the standard Smagorinsky model underpredicts the instantaneous SGS stress tensor magnitudes, one worries that it also underpredicts the mean SGS stress, i.e., the net momentum flux associated with the subgrid scales. For flows with mean shear and when the filter scale is not negligibly small compared to the integral scale of turbulence, the subgrid scales can carry significant amounts of mean momentum flux and thus such underprediction can be a serious issue for LES.

In this Brief Communication, we present an analysis of the mean momentum flux predicted by the Smagorinsky and mixed models in flows with mean shear. The condition that the Smagorinsky model predict the correct SGS mean momentum flux can be written as

$$\langle \tau_{ij}^d\rangle = -2c_s^2\Delta^2\langle(2\tilde{S}_{mn}\tilde{S}_{mn})^{1/2}\tilde{S}_{ij}\rangle, \tag{1}$$

where the superscript d refers to the deviatoric part of the tensor. For isotropic turbulence $\langle\tau_{ij}^d\rangle=0$ and the mean shear condition does not provide any useful information about the model. Conversely, consider a homogeneous turbulent flow with uniform mean velocity gradient tensor $\langle A_{ik}\rangle=\partial\langle u_i\rangle/\partial x_k$. For such a flow, $\langle\tau_{ij}^d\rangle$ equals

$$\langle\tau_{ij}^d\rangle = \int\int_{-\infty}^{\infty}\int (1-|\hat{G}_\Delta(\mathbf{k})|^2)\Theta_{ij}^d d^3\mathbf{k}, \tag{2}$$

where $\hat{G}_\Delta(\mathbf{k})$ is the transfer function of the filter at scale Δ, $\mathbf{k}$ is the wavevector, and Θ_{ij} is the spectral energy density tensor. Starting with Lumley,[9] there is growing evidence[10–12] that the co-spectrum of turbulence with mean shear displays universal features in the inertial range. Recently, Ishihara *et al.*[12] have proposed and tested a functional form which is general under the condition that Θ_{ij} depends linearly upon the mean velocity gradient tensor $\langle A_{mn}\rangle$:

$$\Theta_{ij}(\mathbf{k}) = \frac{c_K}{4\pi}\epsilon^{2/3}k^{-11/3}P_{ij}(\boldsymbol{\kappa}) + \epsilon^{1/3}k^{-13/3}T_{ij\alpha\beta}(\boldsymbol{\kappa})\langle A_{\alpha\beta}\rangle, \tag{3}$$

where $P_{ij}(\boldsymbol{\kappa})=\delta_{ij}-\kappa_i\kappa_j$ is the standard projection tensor, $\kappa_i=k_i/k$, and

$$T_{ij\alpha\beta}(\boldsymbol{\kappa}) = A[P_{i\alpha}(\boldsymbol{\kappa})P_{j\beta}(\boldsymbol{\kappa}) + P_{i\beta}(\boldsymbol{\kappa})P_{j\alpha}(\boldsymbol{\kappa})] + BP_{ij}(\boldsymbol{\kappa})\kappa_\alpha\kappa_\beta. \tag{4}$$

From analysis of a direct numerical simulation (DNS) database, Ishihara *et al.*[12] determine that $A\sim -0.16$, $B\sim -0.4$ and show that this gives good agreement with the Lumley co-spectrum.[9]

Using this spectral model, we may now evaluate the integral of Eq. (2) (from here on, τ_{ij} will denote the deviatoric part for notational simplicity):

$$\langle\tau_{ij}\rangle = \int_0^{+\infty}(1-|\hat{G}_\Delta(\mathbf{k})|^2)\frac{c_K}{4\pi}\epsilon^{2/3}k^{-5/3}dk\; Y_{ij} + \int_0^{+\infty}(1-|\hat{G}_\Delta(\mathbf{k})|^2)\langle A_{\alpha\beta}\rangle\epsilon^{1/3}k^{-7/3}dk\; Z_{ij\alpha\beta}, \tag{5}$$

where $Y_{ij}\equiv\oint[P_{ij}(\boldsymbol{\kappa})-\frac{1}{3}\delta_{ij}P_{mm}(\boldsymbol{\kappa})]\,d\Omega$, $Z_{ij\alpha\beta}\equiv\oint[T_{ij\alpha\beta}(\boldsymbol{\kappa})-\frac{1}{3}\delta_{ij}T_{mm\alpha\beta}(\boldsymbol{\kappa})]\,d\Omega$ with Ω being solid angle, and obtain

$$\langle\tau_{ij}\rangle = \tfrac{8}{15}\pi^{-1/3}(7A-B)\epsilon^{1/3}\Delta^{4/3}\mathcal{I}_s\left(-\tfrac{7}{3}\right)\langle S_{ij}\rangle, \tag{6}$$

where $\mathcal{I}_s(x)=\int_0^\infty\kappa^x(1-|\hat{G}_\pi|^2)d\kappa$. Due to the linear relationship of the spectral tensor with the mean velocity gradient and because $T_{ij\alpha\beta}$ is symmetric in α and β, this result is proportional to the mean strain-rate tensor.

The mean stress predicted by the Smagorinsky model is

$$\begin{aligned}\langle\tau_{ij}^S\rangle &= -2c_s^2\Delta^2\langle|\tilde{S}|\tilde{S}_{ij}\rangle\\ &\approx -2c_s^2\Delta^2\langle 2\tilde{S}_{mn}^2\rangle^{1/2}\langle S_{ij}\rangle\\ &= -2^{3/2}c_s^2\Delta^2\left[\langle S_{mn}\rangle^2 + c_K\epsilon^{2/3}\left(\frac{\pi}{\Delta}\right)^{4/3}\mathcal{I}\left(\frac{1}{3}\right)\right]^{1/2}\langle S_{ij}\rangle,\end{aligned} \tag{7}$$

where $\mathcal{I}(x)=\int_0^\infty\kappa^x|\hat{G}_\pi|^2d\kappa$. Equating $\langle\tau_{ij}\rangle$ with $\langle\tau_{ij}^S\rangle$ and solving for the Smagorinsky coefficient yields

$$(c_s^2)_{\text{mom}} = \frac{\dfrac{2\sqrt{2}}{15}(B-7A)c_K^{-1/2}\pi^{-1}\mathcal{I}_s\left(-\dfrac{7}{3}\right)\mathcal{I}^{-1/2}\left(\dfrac{1}{3}\right)}{\left[1+\left(2c_K\mathcal{I}\left(\dfrac{1}{3}\right)\right)^{-1}\pi^{-4/3}S^{*2}\right]^{1/2}}, \tag{8}$$

where $S^*=(2\langle S_{mn}\rangle\langle S_{mn}\rangle)^{1/2}\epsilon^{-1/3}\Delta^{2/3}$ is the mean strain magnitude normalized with the inertial-range time at scale Δ. Evaluating the integrals $\mathcal{I}$ and $\mathcal{I}_s$ for the cut-off filter, and replacing with $c_K=1.6$, $A=-0.16$ and $B=-0.4$, yields

$$(c_s)_{\text{mom}} = 0.17(1+0.091S^{*2})^{-1/4}. \tag{9}$$

In the limit of small mean strain S^* the result $(c_s)_{\text{mom}}\approx 0.17$ is essentially equal to the value obtained from the dissipation balance (up to two significant numbers). Note that the condition of momentum balance, even in the limit $S^*\to 0$, yields a different expression for the coefficient than dissipation balance. To see this clearly, note from Eq. (8) that $(c_s)_{\text{mom}}(S^*\to 0)$ depends on the constants A, B, and c_K, whereas c_s only depends upon c_K. Note also that the present result for $(c_s)_{\text{mom}}$ differs from what is obtained from dissipation balance in cases where the effects of mean shear are explicitly taken into account, such as the model calculation of Yoshizawa.[13]

For the Gaussian filter, we obtain

$$(c_s)_{\text{mom}} = 0.26(1+0.088S^{*2})^{-1/4}. \tag{10}$$

When $S^*\to 0$, $(c_s)_{\text{mom}}=0.26$, which now is significantly larger than $c_s=[2c_K\mathcal{I}(\frac{1}{3})]^{-3/4}\pi^{-1}\approx 0.16$ obtained from dissipation balance for the Gaussian filter. This difference would imply that the Smagorinsky model with the standard coefficient obtained from dissipation balance would underestimate the mean SGS momentum fluxes by a factor $(0.16/0.26)^2\sim 0.38$. Nevertheless, it is well known[6,14,15] that for the Gaussian filter one must also consider the Leonard stress as part of the model. For our purposes it is convenient to work with the Taylor-series approximation of the model, also called the nonlinear, or tensor eddy-viscosity model,[2,16,17] written as

$$\tau_{ij}^{SL} = -2c_s^2\Delta^2|\tilde{S}|\tilde{S}_{ij} + \frac{\Delta^2}{12}\left(N_{ij}-\frac{1}{3}\delta_{ij}N_{mm}\right), \tag{11}$$

where $N_{ij}\equiv(\partial\tilde{u}_i/\partial x_k)(\partial\tilde{u}_j/\partial x_k)$. To evaluate the mean momentum flux predicted from this model, the average of N_{ij} is required, which can be evaluated from the spectral tensor. One thus obtains

$$\begin{aligned}\langle\tau_{ij}^{SL}\rangle &= -2^{3/2}c_s^2\Delta^2\left[\langle S_{mn}\rangle^2 + c_K\epsilon^{2/3}\Delta^{-4/3}\pi^{4/3}\mathcal{I}\left(\frac{1}{3}\right)\right]^{1/2}\langle S_{ij}\rangle\\ &+ \frac{2}{45}(7A-B)\pi^{5/3}\epsilon^{1/3}\Delta^{4/3}\mathcal{I}\left(-\frac{1}{3}\right)\langle S_{ij}\rangle\\ &+ \frac{\Delta^2}{12}\left[\langle A_{ik}\rangle\langle A_{jk}\rangle - \frac{1}{3}\delta_{ij}\langle A_{mk}\rangle\langle A_{mk}\rangle\right].\end{aligned} \tag{12}$$

This model now includes a term which is not proportional to the mean strain rate. To equate this predicted mean stress with the one obtained from the SGS spectrum [Eq. (6), which is proportional to the mean strain rate] we choose to equate the predicted projections onto the mean strain-rate tensor, i.e., $\langle\tau_{ij}\rangle\langle S_{ij}\rangle=\langle\tau_{ij}^{SL}\rangle\langle S_{ij}\rangle$. Solving for the coefficient yields

$$\begin{aligned}(c_{SL}^2)_{\text{mom}} &= \left[1+\frac{1}{2}\left(c_K\mathcal{I}\left(\frac{1}{3}\right)\right)^{-1}\pi^{-4/3}S^{*2}\right]^{-1/2}\\ &\times\left\{\frac{\Gamma}{48}c_K^{-1/2}\mathcal{I}^{-1/2}\left(\frac{1}{3}\right)\pi^{-2/3}S^*\right.\\ &+ (B-7A)c_K^{-1/2}\mathcal{I}^{-1/2}\left(\frac{1}{3}\right)\left[\frac{2\sqrt{2}}{15}\pi^{-1}\mathcal{I}_s\left(-\frac{7}{3}\right)\right.\\ &\left.\left.-\frac{\sqrt{2}}{90}\pi\mathcal{I}\left(-\frac{1}{3}\right)\right]\right\},\end{aligned} \tag{13}$$

where $\Gamma=\langle A_{ik}\rangle\langle A_{jk}\rangle\langle A_{ij}\rangle(\langle S_{mn}\rangle\langle S_{mn}\rangle)^{-3/2}$ is a parameter characterizing the structure of the mean velocity gradient. For planar shear, $\Gamma=0$, for axisymmetric extension, $\Gamma=-\sqrt{6}/6$, and $\Gamma=\sqrt{6}/6$ for axisymmetric contraction. Evaluating numerically, in the limit of small S^* we obtain $(c_{SL})_{\text{mom}}\approx 0.15$, independently of Γ.

The energy balance for the Smagorinsky model supplemented with the nonlinear model requires us to calculate $\langle N_{ij}\widetilde{S}_{ij}\rangle$, a third-order moment that cannot be evaluated from the spectral tensor, but that may be expressed in terms of the skewness coefficient of the filtered velocity-derivatives $\Sigma \equiv \langle(\partial\widetilde{u}_1'/\partial x_1)^3\rangle/\langle(\partial\widetilde{u}_1'/\partial x_1)^2\rangle^{3/2}$ in the limit of small S^*, where it appears reasonable to assume isotropic statistics of the resolved third-order moments. Also, measurements[18] show $\Sigma \approx -0.4$ for the Gaussian filter. Evaluating the coefficient in the limit of small S^* yields

$$c_{SL}=\left\{\left[2c_K\mathcal{I}\left(\frac{1}{3}\right)\pi^{4/3}\right]^{-3/2}+\frac{7\sqrt{15}}{1080}\Sigma\right\}^{1/2}\approx 0.13. \tag{14}$$

Thus for the Gaussian filter, when the Smagorinsky model is supplemented with the Leonard-type term, the momentum balance again yields results quite close to those of the energy dissipation balance.

Finally, one may use the two conditions of dissipation and momentum balance to simultaneously solve for two coefficients in a mixed model in which the nonlinear model is written with an arbitrary coefficient, i.e.,

$$\tau_{ij}^{SM}=-2C_1\Delta^2|\widetilde{S}|\widetilde{S}_{ij}+C_2\Delta^2\left(N_{ij}-\tfrac{1}{3}N_{mm}\delta_{ij}\right). \tag{15}$$

The two conditions are $\epsilon=-\langle\tau_{ij}^{SM}\widetilde{S}_{ij}\rangle$ and $\langle\tau_{ij}\rangle\langle S_{ij}\rangle=\langle\tau_{ij}^{SM}\rangle \times\langle S_{ij}\rangle$. The approach of using two different balance conditions to solve for two independent coefficients was utilized for balancing dissipations of energy and enstrophy.[18] Replacing and evaluating the expressions yields the pair of equations

$$a_\epsilon C_1+b_\epsilon C_2=c_\epsilon,\quad a_\tau C_1+b_\tau C_2=c_\tau, \tag{16}$$

where

$$c_\epsilon=1,\quad a_\epsilon=[S^{*2}+2c_K\mathcal{I}(1/3)\pi^{4/3}]^{3/2}, \tag{17}$$

$$b_\epsilon=-\frac{\sqrt{2}}{4}\Gamma S^{*3}-\frac{4}{15}(5A+B)\pi^{5/3}\mathcal{I}\left(-\frac{1}{3}\right)S^{*2} -\frac{7\sqrt{30}}{45}[c_K\mathcal{I}(1/3)]^{3/2}\pi^2\Sigma, \tag{18}$$

$$c_\tau=\frac{4}{15}(7A-B)\pi^{-1/3}\mathcal{I}_s(-7/3), \tag{19}$$

$$a_\tau=-[S^{*2}+2c_K\mathcal{I}(1/3)\pi^{4/3}]^{1/2}, \tag{20}$$

$$b_\tau=\frac{\sqrt{2}}{4}\Gamma S^*+\frac{4}{15}(7A-B)\mathcal{I}\left(-\frac{1}{3}\right)\pi^{5/3}. \tag{21}$$

Evaluating for the cut-off filter and using $\Sigma\sim -0.3$ (valid for the cut-off filter[18]) we obtain

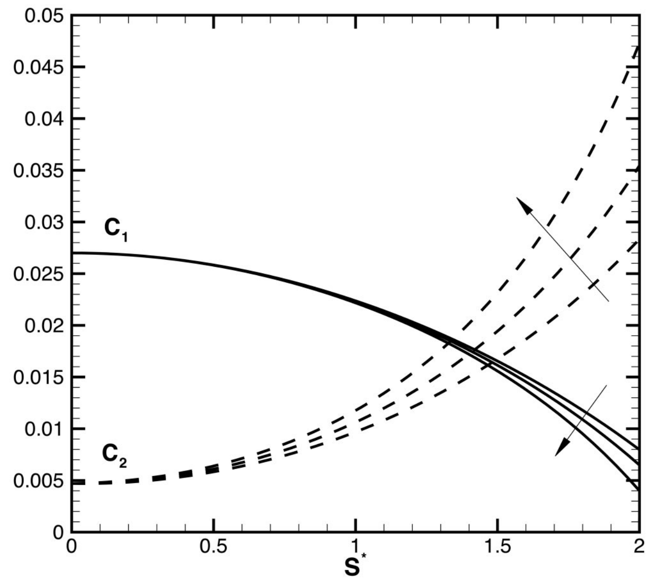

FIG. 1. C_1 and C_2 calculated with simultaneous energy and momentum balance for cut-off filter, with $\Gamma=-\sqrt{6}/6, 0$, and $\sqrt{6}/6$. Solid lines: C_1; dashed lines: C_2. The arrows mark the directions along which Γ increases.

$$C_1=0.027\frac{F_1(S^*)}{F_2(S^*)},\quad\text{and}\quad C_2=0.0047\frac{F_3(S^*)}{F_4(S^*)}, \tag{22}$$

where $F_1(S^*)=\sqrt{1+0.091S^{*2}}(1-0.22\Gamma S^*-0.20S^{*2}+0.022\Gamma S^{*3})$, $F_2(S^*)=1-0.22\Gamma S^*+0.019S^{*2}-0.020\Gamma S^{*3}-0.0065S^{*4}+1.7\times 10^{-8}\Gamma S^{*5}$, $F_3(S^*)=1+1.1S^{*2}$ and $F_4(S^*)=1-0.22\Gamma S^*-0.071S^{*2}+1.8\times 10^{-7}\Gamma S^{*3}$. The results are plotted in Fig. 1 as function of S^*, for the three extreme values of Γ. When the same procedure is applied for the Gaussian filter, one obtains

$$C_1=0.014\frac{F_5(S^*)}{F_6(S^*)},\quad\text{and}\quad C_2=0.099\frac{F_7(S^*)}{F_8(S^*)}, \tag{23}$$

where $F_5(S^*)=\sqrt{1+0.088S^{*2}}(1-0.45\Gamma S^*-0.91S^{*2}+0.10\Gamma S^{*3})$, $F_6(S^*)=1-0.24\Gamma S^*+0.013S^{*2}-0.022\Gamma S^{*3}-0.0066S^{*4}-2.2\times 10^{-10}\Gamma S^{*5}$, $F_7(S^*)=1+0.14S^{*2}$ and $F_8(S^*)=1-0.24\Gamma S^*-0.075S^{*2}-2.5\times 10^{-9}\Gamma S^{*3}$. The results are plotted in Fig. 2. We can see that for both filter types, the nonlinear model part becomes more important at large nondimensional mean velocity gradient S^*. At high S^*, C_1 even becomes negative (backscatter).

In conclusion, a recent model spectrum[12] for homogeneous turbulence with mean velocity gradients can be used to evaluate the mean momentum fluxes caused by the subgrid scales. This allows quantifying SGS model coefficients based on mean SGS momentum flux balance instead of the traditional approach of using kinetic energy dissipation. Remarkably, the Smagorinsky coefficient deduced from momentum balance is very close to the traditional coefficient deduced from mean dissipation balance. An analysis of the mixed model where two coefficients are adjusted to obtain jointly the proper balance of kinetic energy and mean momentum yields a value that increases with mean shear for the coefficient of the nonlinear-model part.

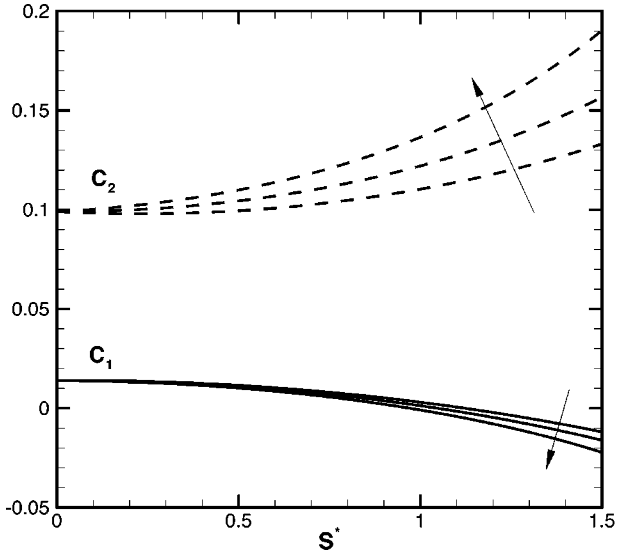

FIG. 2. Same as Fig. 1, but for Gaussian filter.

Our results show that the traditional Smagorinsky model should predict correctly the mean SGS stress in flows with weak mean velocity gradients. This conclusion differs from the observations that the Smagorinsky model tends to underpredict the magnitude of the instantaneous, fluctuating SGS stresses.[3,7,8] A corollary to this conclusion is that if it is observed that the Smagorinsky model does not predict the correct mean SGS stress (as is known to happen near solid walls, for example), the cause must lie in a failure of the model spectrum [Eq. (3)] used here. Such a failure can occur at large scales when Δ approaches the integral scale, or when the magnitude of the mean velocity gradient is sufficiently large to void the linearity assumption upon which Eq. (3) is based. In that case, higher-order terms would need to be added to Eq. (3), such as $k^{-15/3}Q_{ijmqnp}(\boldsymbol{\kappa})\,\partial\langle u_m\rangle/\partial x_q\,\partial\langle u_n\rangle/\partial x_p$, where $Q_{ijmqnp}(\boldsymbol{\kappa})$ is the isotropic 6th rank tensor function of a unit vector $\boldsymbol{\kappa}$. Note also that since $T_{ij\alpha\beta}$ in Eq. (3) is symmetric in both (ij) and $(\alpha\beta)$, predictions for mean SGS stress obtained in this paper do not depend upon mean rotation. Only at second order will such dependency occur. Unfortunately, to fully characterize $Q_{ijmqnp}(\boldsymbol{\kappa})$, a total of 17 scalar parameters are required. The task of finding these parameters from empirical data would be challenging.

This work has been performed with financial support from NSF Grant No. CTS-0120317.

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